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ROTATION AND MAGNETIC FIELD  
ON THE DRAG ON A SPHERE  
OSCILLATING IN A CONDUCTING,  
VISCOUS MEDIUM**

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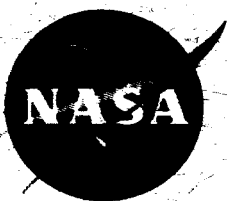
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**GODDARD SPACE FLIGHT CENTER**

**GREENBELT, MARYLAND**

Effect of Compressibility, Rotation and Magnetic Field  
on the Drag on a Sphere Oscillating in a Conducting,  
Viscous Medium

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I. Introduction

The role of rotation and magnetic fields in cosmical fluid dynamics is well known. Recently, some attention has been given by several authors [1-7] on the interaction of Coriolis forces and Lorentz forces on flow phenomena.

It is generally accepted [1,2] that hydromagnetic flow in the earth's liquid core is somehow responsible for the main geomagnetic field, and therefore a theory of the dynamics of core motions is required in order to understand a plausible theory of the earth's magnetic field. It has been suggested that the near coincidence between the geographic and geomagnetic poles is the result of the strong influence of Coriolis forces, due to the earth's rotation, on motions in the core.

So much of meteorology depends ultimately upon the dynamics of a revolving fluid. The large-scale and moderate motions of the atmosphere are greatly affected by the rotation of the earth. Kelvin is said

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to have pointed out [8] that rotation confers on a fluid certain properties resembling those of an elastic solid, and a rotating fluid can transmit waves. In the case of an infinite liquid, rotating as a rigid body about an axis, the amount of energy possessed by the liquid is infinite and it is of great interest to know how small disturbances propagate in such a liquid.

To understand some of these phenomena, it will be interesting to study the flow of a rotating fluid around elementary bodies.

We consider here small oscillations of a sphere in a compressible, viscous, electrically conducting and rotating medium in the presence of a uniform magnetic field. In addition to the above mentioned interests, this problem may have some applications in connection with the transmission of sound by fog [see 9, p. 659].

The classical problem of the oscillation of a sphere in a viscous fluid was first considered by Stokes [9, p. 643]. The drag force experienced by the sphere is (in dimensionless form)\*

$$X_0 = -6 \pi u_s \left( \frac{1}{R_e} + \frac{1}{\sqrt{2R_e}} \right) - 2 \pi \frac{d u_s}{d t} \left( \frac{1}{3} + \frac{3}{\sqrt{2R_e}} \right) . \quad (1)$$

It is the purpose of the present investigation to determine the effects of the magnetic field, compressibility, coriolis forces and their

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\*The dimensionless quantities and parameters in (1) are explained in section II.

mutual interaction on the drag and other physical variables. Because of the motion of the fluid in the magnetic field, an associated electrical field is produced which, according to Ohm's law sets up electrical currents in the fluid if the latter is a conductor. The interaction of these currents with the magnetic field then produces a body force which must be included in the Navier-Stokes equations for the motion of the fluid. The effect of this body force is to inhibit the motion of the fluid across the lines of force. The viscous effect gives rise to viscous dissipative waves, the magnetic effect is responsible for Alfvén waves while the compressibility and rotation produce sound and the so-called Taylor waves respectively. Their mutual interaction will make the situation even more complicated. In the next section, we make several assumptions to make the problem mathematically more amenable.

## II. Fundamental Equations and Formulation of the Problem

Let the sphere oscillate along the axis of the rotating fluid which is taken to be the x-axis. The origin is at the mean position of the center of the sphere. Let the uniform magnetic field be  $H^0 \underline{i}$ . The equations of motion of a compressible, viscous, unbounded fluid rotating with a constant angular velocity  $\underline{\Omega}$ , referred to a rotating frame of reference are (in m.k.s. units)

$$\begin{aligned} \rho \frac{d \underline{v}}{d t} + 2 \rho \underline{\Omega} \times \underline{v} + \rho \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = - \nabla p + \mu \operatorname{curl} \underline{H} \times \underline{H} + \\ + \rho \nu \nabla^2 \underline{v} + \frac{\rho \nu}{3} \nabla \operatorname{div} \underline{v} , \end{aligned} \quad (2)$$

where  $\rho$  is the density,  $\underline{v}$  the velocity of the fluid,  $\underline{H}$  is total magnetic field,  $p$  is the pressure,  $r$  is the radial coordinate:  $r^2 = y^2 + z^2$ ,  $\nu$  the kinematic viscosity (assumed constant) and  $\mu$  the magnetic permeability (assumed constant).

The equation of continuity is

$$\operatorname{div} (\rho \underline{v}) + \frac{\partial \rho}{\partial t} = 0 . \quad (3)$$

The physical equation (the conduction of heat is neglected) is

$$\frac{1}{p} \frac{d p}{d t} = \frac{\gamma}{\rho} \frac{d \rho}{d t} ,$$

(where  $\gamma$  denotes the ratio of two specific heats) which can be written as [9, p. 654]

$$p = p^0 + \rho^0 c^2 s, \quad (4)$$

where  $c$  is the velocity of sound in the absence of viscosity,  $p^0$ ,  $\rho^0$  represent pressure and density in the undisturbed state and  $s$  denotes the condensation

$$\rho = \rho^0 (1 + s) . \quad (5)$$

Maxwell's equations with the usual notation for electromagnetic quantities are

$$\begin{aligned} (i) \quad \text{curl } \underline{H} &= \underline{J}, & (ii) \quad \text{div } \underline{H} &= 0, \\ (iii) \quad \text{curl } \underline{E} &= -\mu \frac{\partial \underline{H}}{\partial t}, & (iv) \quad \text{div } \underline{E} &= 0, \\ (v) \quad \underline{J} &= \sigma [\underline{E} + \mu \underline{v} \times \underline{H}], \end{aligned} \quad (6)$$

where the electrical conductivity  $\sigma$  is treated as constant.

From (i), (iii) and (v) we obtain

$$\frac{\partial \underline{H}}{\partial t} - \text{curl } (\underline{v} \times \underline{H}) = \frac{1}{\mu \sigma} \nabla^2 \underline{H} . \quad (7)$$

As in the classical problem, the convective terms in (2) will henceforth be neglected. This is justified if

$$R_s = a |v_s|/\nu \ll 1, \quad (8)$$

where  $v_s$  is the velocity of the sphere and  $a$  is its radius.

Further linearization of (2) and (3) is possible by means of (5) which yields

$$\begin{aligned} \rho^0 \frac{\partial \underline{v}}{\partial t} + 2 \rho^0 \underline{\Omega} \times \underline{v} + \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) \rho^0 = & - \nabla p + \mu \operatorname{curl} \underline{H} \times \underline{H} + \\ & + \rho^0 \nu \nabla^2 \underline{v} + \frac{\rho^0 \nu}{3} \nabla \operatorname{div} \underline{v} \end{aligned} \quad (9)$$

and

$$\operatorname{div} \underline{v} + \frac{\partial s}{\partial t} = 0. \quad (10)$$

These equations will be now reduced to a dimensionless form by referring the length to  $a$ , the radius of the sphere, the velocity to  $a\lambda$  ( $\lambda/2\pi$  is the frequency of the oscillation), the time to  $1/\lambda$ , the magnetic field to  $H^0$ , the electric field to  $\mu a \lambda H^0$ , the current density  $\underline{J}$  to  $H^0/a$  and the pressure to  $\rho^0 a^2 \lambda^2$ . The equations (6), (7) then become

$$\begin{aligned}
 (i) \quad \text{curl } \underline{H} &= \underline{J}, & (ii) \quad \text{div } \underline{H} &= 0, \\
 (iii) \quad \text{curl } \underline{E} &= - \frac{\partial \underline{H}}{\partial t}, & (iv) \quad \text{div } \underline{E} &= 0, \\
 (v) \quad \text{curl } \underline{H} &= R_m [\underline{E} + \underline{v} \times \underline{H}]
 \end{aligned} \tag{11}$$

and

$$\nabla^2 \underline{H} = R_m \left[ \frac{\partial \underline{H}}{\partial t} - \text{curl } (\underline{v} \times \underline{H}) \right] \tag{12}$$

where

$$R_m \text{ (Magnetic Reynolds number)} = a^2 \lambda \mu \sigma . \tag{13}$$

In addition, we introduce the following parameters:

$$R_e \text{ (Reynolds number)*} = a^2 \lambda / \nu \tag{14a}$$

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\*It should be noted that we have used  $a\lambda$  as velocity in the definition of the Reynolds number. There still is another important dimensionless parameter  $\alpha/a$ , where  $\alpha$  is the amplitude of the oscillatory motion of the sphere. It is assumed that this quantity is small compared to unity.

The velocity of sphere is  $v_s \underline{i} = \alpha \lambda e^{i\lambda t} \underline{i}$ , or in dimensionless form,  $u_s \underline{i} = \alpha/a e^{it} \underline{i}$ .

The Reynolds number for the flow might also have been defined as

$$R_s = a \alpha \lambda / \nu \quad (=R_e \alpha/a).$$

It is necessary that

$$R_s \ll 1 ,$$

whereas the requirement for  $R_e$  is less severe.



$$\beta \text{ (magnetic pressure number)} = \frac{\mu H_0^2}{\rho^* a^2 \lambda^2}, \quad (14b)$$

the ratio of the magnetic pressure to dynamic pressure, and

$$\delta = 2 \Omega / \lambda, \quad (14c)$$

the inverse of the so-called Rossby number  $R_0 = \lambda / (2\Omega) = 1/\delta$ . In terms of these parameters, equations (9) and (10) reduce to

$$\frac{\partial \underline{v}}{\partial t} + \delta \underline{i} \times \underline{v} = \beta \text{curl } \underline{H} \times \underline{H} + \frac{1}{R_e} \nabla^2 \underline{v} - (c^2 + \frac{1}{3} R_e) \nabla s \quad (15)$$

and

$$\text{div } \underline{v} = - \frac{\partial s}{\partial t} \quad (16)$$

where velocity of sound  $c$  is now in non-dimensional form referred to  $a\lambda$ , and if we assume that

$$R_0 \gg 1, \quad (17)$$

so as to neglect second and higher powers of  $\delta$ , the centrifugal force in (9) being of  $O(\delta^2)$  is dropped\* in (15).

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\*Notice that for large values of radial distance, our neglecting centrifugal force may not be justified even though  $\delta^2$  is negligible. In the Incompressible case, it presents no difficulty since it can be absorbed with pressure term. However, in Compressible case, this cannot be done. But since we are primarily interested in the effects of Coriolis forces on the motion, following the standard procedure in such cases (see for example Hide and Roberts 1960 [2] and Talwar 1964 [5]), centrifugal force may be ignored.

The boundary conditions on  $\underline{v}$  are  $\underline{v} = u_s \underline{i}$  on the surface of the sphere and is zero at infinity. If the conductivity  $\sigma'$  of the sphere\* is comparable to the conductivity  $\sigma$  of the fluid, we require that  $\underline{H}$  should be continuous across the surface of the sphere, while at infinity,  $\underline{H} = \underline{i}$ .\*\*

We next make two important assumptions:

(1) The magnetic Reynolds number  $R_m$  is a small parameter.

We can then find a perturbation solution for small  $R_m$ . This is a valid assumption in most practical problems and this technique has been adopted in several discussions (see for example, Ludford [10] and Tamada [11]).

(2) The magnetic pressure is comparable to dynamic pressure, i.e. we assume  $\beta = O(1)$ .

We assume an expansion in powers of  $R_m$ :

$$\begin{aligned}\underline{v} &= \underline{v}_0 + R_m \underline{v}_1 + \dots \\ \underline{H} &= \underline{H}_0 + R_m \underline{H}_1 + \dots \\ \underline{E} &= \underline{E}_0 + R_m \underline{E}_1 + \dots \\ s &= s_0 + R_m s_1 + \dots\end{aligned}\tag{18}$$

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\*The discussion will be valid even if the sphere is an insulator. We assume that the permeabilities of the sphere and the fluid are equal.

\*\*These conditions imply that the tangential components of electric field will be continuous across the sphere and that  $\underline{E} = 0$  at infinity.

and obtain a set of equations from (11), (12), (15) and (16) with appropriate boundary conditions. It should be noted that some of the complications which arise from boundary conditions in magneto-gas-dynamics are conveniently absent in this problem.

### III. Solution of the Resulting Equations

#### Zeroth Order Equations

(i)  $\text{curl } \underline{H}_0 = 0$ ,  $\text{div } \underline{H}_0 = 0$ , that is the magnetic field to this order is independent of the fluid velocity. The same equations hold inside the sphere and the boundary conditions are that  $\underline{H}_0 = \underline{i}$  at infinity and it is continuous across the surface of the sphere. The solution is, therefore,

$$\underline{H}_0 \equiv \underline{i} . \quad (19)$$

(ii) The electric field  $\underline{E}_0$  satisfies  $\text{curl } \underline{E}_0 = 0$ ,  $\text{div } \underline{E}_0 = 0$ , both in the fluid and the sphere. The continuity of the tangential electric field on the surface of the sphere and vanishing of  $\underline{E}_0$  at infinity implies

$$\underline{E}_0 \equiv 0 . \quad (20)$$

(iii) The zeroth order flow fields and pressure fields are unaffected by the magnetic field and are given by\*

$$(\nabla^2 + h^2)\underline{v}_0 = -i\left(\frac{k^2 - h^2}{k^2}\right) \nabla s_0 - \delta R_e \underline{v}_0 \times \underline{i} \quad (21)$$

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\*The boundary condition  $\underline{v} = u_s \underline{i} = \frac{\alpha}{a} e^{it} \underline{i}$  on the surface of the sphere and the linearity of the system of differential equations imply that the only time dependence of the physical variables is a factor  $e^{it}$ .

and

$$\operatorname{div} \underline{v}_0 = -i s_0 \quad (22)$$

where

$$k^2 = \frac{3 R_e}{4 i + 3 R_e} c^2 \quad (23)$$

$$h^2 = -i R_e. \quad (24)$$

As in the non-rotating case, we have here axial symmetry in that the physical variables are independent of the azimuthal coordinate  $\varphi$ .<sup>\*</sup> However, the azimuthal components of velocity and magnetic field will be non-zero here in contrast to the non-rotating case in which  $v_\varphi$  and  $H_\varphi$  were zero.

From (21) and (22) it follows that  $v_{c\chi} = O(1)$ ,  $v_{or} = O(1)$  and  $v_{o\varphi} = O(\delta)$  and hence

$$(\nabla^2 + k^2) s_0 = O(\delta^2). \quad (25)$$

It follows that the coupled equations (21) can be simplified in this case. In fact, in view of (17) and (25),  $v_{c\chi}$  and  $v_{or}$  satisfy identically

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<sup>\*</sup>( $r, \varphi, \chi$ ) are the cylindrical coordinates.

the same system of equations and boundary conditions as in the non-rotating case. The solution is therefore [9, p. 656]

$$\begin{aligned} \text{(i)} \quad s_o &= A \times f_1(k R) \\ \text{(ii)} \quad \underline{v}_o &= \frac{i}{k^2} \nabla s_o + \underline{v}_o^c \end{aligned} \tag{26}$$

where the first term in (ii) is a particular integral of (21) and (22) and the last term

$$\underline{v}_o^c = B[2f_o(hR) \nabla \times - h^2 R^5 f_2(hR) \nabla \frac{x}{R^3}] , \tag{27}$$

is the complementary function.  $(R, \theta, \varphi)^*$  are the spherical polar coordinates. The function  $f_n(\zeta)$ , defined as

$$f_n(\zeta) = \left(-\frac{1}{\zeta} \frac{d}{d\zeta}\right)^n \frac{e^{-i\zeta}}{\zeta} , \tag{28}$$

represents spherical waves with rapidly diminishing amplitude--the one with argument  $\zeta = h R$  denotes boundary layer effects (viscous) while the one with argument  $\zeta = k R$  essentially represents compressibility effects (sound waves). The constants A and B are given by

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\*  $x = R \cos \theta$ ,  $r = R \sin \theta$ .

$$\begin{aligned}
 A &= \frac{-i k^2 h f_1'(h) u_s}{2k f_1(h) f_1'(k) + h f_1'(h) [k f_1'(k) + f_1(k)]} , \\
 B &= \frac{k f_1'(k) u_s}{3[2k f_1(h) f_1'(k) + h f_1'(h) \{k f_1'(k) + f_1(k)\}]} .
 \end{aligned}
 \tag{29}$$

Now  $v_{\text{op}}$  can be determined in a straight-forward manner from (21). In fact, exactly similar arguments can be applied while determining  $\underline{H}_1$  and  $\underline{v}_1$  to show that only azimuthal components  $H_{1\varphi}$  and  $v_{1\varphi}$  are affected by rotation. In view of this decoupling of the system of equations, the rotational effects, which only generate azimuthal components of velocity and magnetic field, are discussed in Appendix and henceforth, we would only consider non-rotating case.

### First Order Equations

(i) The first order magnetic field  $\underline{H}_1$  satisfies

$$\nabla^2 \underline{H}_1 = -i s_0 \underline{1} - \frac{\partial \underline{v}_0}{\partial x} ,$$

$$\text{div } \underline{H}_1 = 0 ,$$

whose solution is

$$\underline{H}_1 = \underline{H}_1^c + \frac{i}{k^2} s_0 \underline{1} + \frac{i}{k^4} \nabla \frac{\partial s_0}{\partial x} + \frac{1}{h^2} \frac{\partial \underline{v}_0^c}{\partial x} \tag{30}$$

where the first term is the complementary function

$$\underline{H}_1^c = C \nabla \frac{P_2(\cos \theta)}{R^3} \quad (30a)$$

and the remaining terms represent a particular integral.

On the other hand, inside the sphere,  $\text{curl curl } \underline{H}_1 = 0$ ,  $\text{div } \underline{H}_1 = 0$ , which shows that the sphere acts like an insulator up to this order.

It follows that

$$\underline{H}_1 = D \nabla [R^2 P_2 (\cos \theta)] \quad (31)$$

inside the sphere. The constants C and D in (30a) and (31) are determined by the requirement that  $\underline{H}_1$  is continuous across the surface of the sphere.

(ii) Finally, the first order flow fields and pressure fields are given by

$$\left. \begin{aligned} (\nabla^2 + h^2) \underline{v}_1 &= \beta R_e \left[ \nabla H_{1x} - \frac{\partial \underline{H}_1}{\partial x} \right] - i \left( \frac{k^2 - h^2}{k^2} \right) \nabla s_1, \\ \text{div } \underline{v}_1 &= -i s_1 \end{aligned} \right\} \quad (32)$$

from which it follows that



$$(\nabla^2 + k^2) s_1 = \beta k^2 \left[ \frac{1}{k^2} \frac{\partial^2 s_0}{\partial x^2} + i s_0 + \frac{\partial v_{ox}^c}{\partial x} \right]^* . \quad (33)$$

We have therefore

$$s_1 = s_1^c + \frac{i\beta}{2} x \frac{\partial s_0}{\partial x} - \frac{Ai\beta}{2} x f_0(kR) + \frac{\beta k^2}{k^2 - h^2} \frac{\partial v_{ox}^c}{\partial x} \quad (34)$$

where the first term

$$s_1^c = E_1 R \cdot P_1(\cos\theta) f_1(kR) + E_2 R^3 \cdot P_3(\cos\theta) f_3(kR) \quad (35)$$

is the complementary function of (33) and the remaining terms are a particular integral; and

$$\begin{aligned} \underline{v}_1 = \underline{v}_1^c - \frac{\beta}{2k^2} \left\{ \nabla \left( x \frac{\partial s_0}{\partial x} \right) - \frac{2}{h^2 - k^2} \nabla \frac{\partial^2 s_0}{\partial x^2} \right\} - \frac{\beta}{k^2} \nabla s_0 + \\ + \frac{\beta h^2}{k^2(h^2 - k^2)} \frac{\partial s_0}{\partial x} \underline{1} + \frac{\beta A}{2k^2} \nabla [x f_0(kR)] + \frac{i}{k^2} \nabla s_1^c - \\ - \frac{i\beta}{2} x \frac{\partial v_o^c}{\partial x} - \frac{i\beta}{2} \frac{k^2 + h^2}{k^2 - h^2} v_{ox}^c \underline{1} . \end{aligned} \quad (36)$$

Here the first term

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$*v_{ox}^c$  is the x-component of the complementary function  $\underline{v}_o^c$  given by (27).

$$\begin{aligned} \underline{v}_1^c = & F_1[2f_0(hR)\nabla x - h^2R^5f_2(hR)\nabla \frac{x}{R^3}] + \\ & + F_2[4f_2(hR)\nabla\{R^3P_3(\cos\theta)\} - 3h^2R^9f_4(hR)\nabla \frac{P_3(\cos\theta)}{R^4}] \end{aligned} \quad (37)$$

is the complimentary function of (32) and the remaining terms are a particular integral. The constants  $E_1$ ,  $E_2$ ,  $F_1$  and  $F_2$  are determined by the requirement that  $\underline{v}_1 = 0$  at  $R = 1$ .

The following points are worth noting in our solution:

(a) The terms (like  $f_n(hR)$ ) containing the coefficients  $B$ ,  $F_1$  and  $F_2$  represent the compressibility effects on viscous waves.

(b) The terms (like  $f_n(kR)$ ) containing the coefficients  $A$ ,  $E_1$  and  $E_2$  represent viscous effects on sound waves.

(c) The terms corresponding to the coefficients  $C$  and  $D$  represent the potential part of the magnetic field.

At this point, we would discuss in some more detail the case when the fluid is incompressible.

#### IV. Incompressible Case

The solution in this case can be obtained from the above in the limit as  $c \rightarrow \infty$  (i.e.,  $k \rightarrow 0$ ),  $s \rightarrow 0$  and  $c^2 s \rightarrow p$ . (38)

It is clear that we will still have terms (a) and (c) and the corresponding coefficients will be in the limit:

$$\begin{aligned} B &\rightarrow \frac{u_s}{2f_0(h)} \quad , \quad F_1 \rightarrow -\frac{\beta Bi}{10} \left[ 5 + h^2 \frac{f_1(h)}{f_0(h)} \right] \quad , \\ F_2 &\rightarrow -\frac{\beta Bi h^2}{35} [4 + f_1(h)/f_2(h)] \\ C &\rightarrow \frac{h^2 B}{5} [f_2(h) - 2f_3(h)] \quad \text{and} \quad D \rightarrow -\frac{u_s}{10} \quad . \end{aligned} \quad (39)$$

Evidently waves (sound) represented by (b) will no longer be present and the corresponding terms from (b) can be obtained by the limiting process (38):

$$\begin{aligned} p_0 &= c^2 s_0 \rightarrow A_1 \frac{P_1(\cos\theta)}{R^2} \\ p_1 &= c^2 s_1 \rightarrow \left[ \frac{D_1}{R^2} + \beta \left\{ \frac{6}{5} B R f_1(hR) - \frac{2}{5} \frac{A_1 i}{R^2} \right\} \right] P_1(\cos\theta) + \\ &\quad + \left[ \frac{D_2}{R^4} - \beta \left\{ -\frac{6}{5} B h^2 R^3 f_3(hR) + \right. \right. \\ &\quad \left. \left. + \frac{3}{5} \frac{A_1 i}{R^2} - \frac{3C}{R^4} \right\} \right] P_3(\cos\theta) \end{aligned}$$

$$\begin{aligned}
 \underline{v}_0: & \quad \frac{i}{k^2} \nabla s_0 \rightarrow A_1 i \nabla \frac{P_1(\cos\theta)}{R^2} \\
 \underline{v}_1: & \quad - \frac{\beta}{2k^2} \left\{ \nabla \left( x \frac{\partial s_0}{\partial x} \right) - \frac{2}{h^2 - k^2} \nabla \frac{\partial^2 s_0}{\partial x^2} + 2 \nabla s_0 \right. \\
 & \quad \left. - \frac{2h^2}{h^2 - k^2} \frac{\partial s_0}{\partial x} \underline{i} - A \nabla (x f_0(kR)) \right\} + \\
 & \quad + \frac{i}{k^2} \nabla s_1^c \rightarrow i \nabla \left[ D_1 \frac{P_1(\cos\theta)}{R^2} + D_2 \frac{P_3(\cos\theta)}{R^4} \right] + \\
 & \quad + 3 i \beta C \nabla \frac{P_3(\cos\theta)}{R^4} + \frac{\beta}{h^2} \nabla \frac{\partial^2 p_0}{\partial x^2} - \\
 & \quad - \frac{\beta}{10} \frac{\partial}{\partial x} \left[ \frac{1}{R^3} \nabla \left( R^5 \frac{\partial p_0}{\partial x} \right) - R^2 \nabla \frac{\partial p_0}{\partial x} \right] \\
 \underline{H}_1: & \quad \frac{i}{k^2} s_0 \underline{i} + \frac{i}{k^4} \nabla \frac{\partial s_0}{\partial x} \rightarrow \frac{A_1 i}{5} \left[ \frac{1}{R^3} \nabla (R^2 P_2(\cos\theta)) - \right. \\
 & \quad \left. - R^2 \nabla \frac{P_2(\cos\theta)}{R^3} \right]
 \end{aligned} \tag{40}$$

where

$$A_1 = - i h^2 B f_2(h) ,$$

$$D_1 = - \frac{\beta B h^2}{10 f_0(h)} [3 f_1^2(h) - f_0(h) f_2(h)] ,$$

$$D_2 = - \frac{3}{5} \beta B \left[ f_0(h) - 5 f_1(h) + h^2 \cdot \frac{f_1(h) f_3(h)}{f_2(h)} \right] .$$

Next, we can write down the total drag experienced by the sphere\*

$$\begin{aligned}
 X = & - \pi u_s \left[ 6 \left\{ \frac{1}{R_e} + \frac{1}{\sqrt{2R_e}} \right\} + \frac{\beta R_m}{5} \left( 2 + \frac{3}{\sqrt{2R_e}} \right) \right] - \\
 & - \pi \frac{du_s}{dt} \left[ 2 \left\{ \frac{1}{3} + \frac{3}{\sqrt{2R_e}} \right\} - \frac{3\beta R_m}{5\sqrt{2R_e}} \right] .
 \end{aligned} \tag{41}$$

The terms in the curly brackets represent the non-magnetic effect, while the last term in each square bracket represents a hydromagnetic effect. The first square bracket gives a frictional force varying as the velocity: the first term is the Stokes drag for the classical problem in which the sphere undergoes motion of translation; the second term is the oscillatory effect in non-magnetic case; the third term is hydromagnetic, non-viscous, oscillatory effect and the last term is hydromagnetic viscous effect due to oscillation. The magnetic field tends to increase the resistance.

The second square bracket gives the correction to the inertia of the sphere. This amounts to the fraction

$$\frac{1}{2} + \frac{9}{2\sqrt{2R_e}} \left( 1 - \frac{\beta R_m}{10} \right)$$

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\*There will be no contribution of the Maxwell stress to the order we are interested in, as there are no currents on the surface of the sphere:

$$\underline{J} = R_m \left[ \underline{E}_0 + \underline{v}_0 \times \underline{H}_0 \right]_{R=1} = 0 .$$

of the mass of fluid displaced, instead of  $\frac{1}{2}$  as in the frictionless, non-magnetic case. The effect of the magnetic field is to decrease the apparent mass of the fluid.

Although the derivations strictly do not hold for this case, we may consider the value  $R_e \rightarrow \infty$  while at the same time keeping  $R_s \ll 1$ . We then find

$$X = -2\pi \left[ \frac{1}{3} \frac{du_s}{dt} + \frac{1}{5} \beta R_m u_s \right],$$

a result which can be checked in a straight-forward manner from the corresponding hydromagnetic inviscid problem. The non-magnetic part provides the inertia effect and the magnetic part contributes to the frictional part of the drag.

When the period  $\frac{2\pi}{\lambda}$  is made infinitely long, the drag reduces to the Stokes drag in the classical problem in which the sphere moves uniformly in a straight line. This is understandable because as  $\lambda \rightarrow 0$ , the uniform magnetic field  $H^0$  should also approach zero in view of

$$\beta = \frac{\mu H^0{}^2}{\rho a^2 \lambda^2} = O(1)$$

and so the magnetic effect is absent in this limit.

## V. Discussion

First we will discuss the conditions under which our analysis can be expected to give a close approximation to the true flow.

We have neglected the displacement current in comparison with the conduction current, which is justified if

$$\frac{a^2 \lambda^2}{L^2} \ll 1, \quad (42)$$

which follows from the order of magnitudes of the various terms in the Maxwell equation (6, i) where  $L$  is the velocity of light. This is satisfied by all hydromagnetic flow problems in the laboratory. An associated condition, that the excess charge may be neglected, is also satisfied if (42) holds.

As in the classical problem, the convective terms have been neglected. This is justified if\*

$$\alpha \ll a \quad (43)$$

(where  $\alpha$  is the amplitude of the oscillation and  $a$  is the radius of the sphere), provided

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\*This condition justifies the omission of convective terms both in zeroth order and first order approximations as can be seen by the orders of magnitudes of various terms involved in the momentum equation.

$$R_e = \frac{a^2 \lambda}{\nu} = O(1) , \quad (44)$$

which follows from (15), remembering that  $s \rightarrow 0$ ,  $c^2 s \rightarrow p$  in the Incompressible case. For common liquids in hydromagnetic experiments,  $\nu = O(10^{-6})$  in m.k.s. units and hence we must make

$$a^2 \lambda = O(10^{-6}) . \quad (44a)$$

Next, in the zeroth order approximation of the momentum equation, we neglected the Lorentz force  $\underline{J} \times \underline{B} = \beta \text{ curl } \underline{H} \times \underline{H}$ . This is justified if

$$\beta = \frac{\mu H_0^2}{\rho \sigma a^2 \lambda^2} = O(1) \quad (45)$$

and

$$R_m = a^2 \lambda \mu \sigma \ll 1 . \quad (46)$$

In the case of liquid Sodium and Mercury, for example,  $\mu \sigma = O(1)$  and so (46) follows in view of (44a). Condition (45) implies that magnetic pressure is at most comparable to dynamic pressure, which holds for weak field, say 50 gauss or less. In short, our analysis is valid for the oscillations of small globules in the presence of weak magnetic fields.

The following table gives some typical values of these parameters for Mercury, Liquid Sodium and Saturated Salt Water at  $25^\circ\text{C}$  when



$a = .05$  cm,  $B^0 = 25$  gauss,  $\lambda = 25$ .

	Liquid Sodium	Saturated Salt Water At 25° C	Mercury
$R_e$	3	6	60
$\beta$	25	25	2
$R_m$	$11 \times 10^{-5}$	$2 \times 10^{-10}$	$.85 \times 10^{-5}$

Finally, to study the nature and properties of the solution, it will be convenient to introduce the stream functions  $\mathcal{F}$  and  $\mathcal{G}$  (where  $v_x = -\frac{1}{r} \frac{\partial \mathcal{F}}{\partial r}$ ,  $v_r = \frac{1}{r} \frac{\partial \mathcal{F}}{\partial x}$ ,  $H_x = -\frac{1}{r} \frac{\partial \mathcal{G}}{\partial r}$ ,  $H_r = \frac{1}{r} \frac{\partial \mathcal{G}}{\partial x}$ ). For the sake of brevity, we write the results here in a shorter form:

$$\begin{aligned} \mathcal{F} = & e^{it} \sin^2\theta \left[ \frac{a_1(R_e)}{R} + a_2(R_e, R) e^{-(1+i)\sqrt{R_e/2}(R-1)} \right] * \\ & + e^{it} \sin^2\theta \left[ \frac{a_3(R_e)}{R} + \frac{a_4(R_e)}{R^3} + a_5(R_e, R) e^{-(1+i)\sqrt{R_e/2}(R-1)} \right], \\ & + e^{it} \sin^2\theta \cos^2\theta \left[ \frac{a_6(R_e)}{R} + \frac{a_7(R_e)}{R^3} + a_8(R_e, R) e^{-(1+i)\sqrt{R_e/2}(R-1)} \right] \end{aligned} \quad (47)$$

(where the terms under the bar represent the non-magnetic part);

---

\*The wave-like term  $\exp [-(1+i)\sqrt{R_e/2}(R-1)]$  comes from the functions  $f_n(hR)$  defined by (28).

$$\Phi = e^{it} \cos\theta \sin^2\theta \left[ b_1(R_e) + \frac{b_2(R_e)}{R^2} + b_3(R_e, R) e^{-(1+i)\sqrt{R_e/2}(R-1)} \right] \quad (48)$$

As  $R_e \rightarrow \infty$ ,  $a_1 \rightarrow -\frac{1}{2} \frac{\alpha}{a}$ ,  $a_3 \rightarrow -\frac{3}{20} i \beta R_m \frac{\alpha}{a}$ ,

$$a_4 \rightarrow \frac{3}{20} i \beta R_m \frac{\alpha}{a}, \quad a_6 \rightarrow \frac{3}{4} i \beta R_m \frac{\alpha}{a} \quad (49)$$

$$a_7 \rightarrow -\frac{3}{4} i \beta R_m \frac{\alpha}{a}, \quad b_1 \rightarrow \frac{1}{4} R_m \frac{\alpha}{a}$$

$$\text{and } b_2 \rightarrow -\frac{3}{20} R_m \frac{\alpha}{a}.$$

If  $\Phi_s$  is the corresponding function for the magnetic lines of force inside the sphere,

$$\Phi_s = \frac{1}{10} \frac{\alpha}{a} R_m R^3 \cos\theta \sin^2\theta e^{it}, \quad (50)$$

which is current-free and hence up to this approximation, the sphere acts like an insulator.

From (47) and (48) it follows that there are three types of terms in the solution:

(1) Wave-like (corresponding to the appearance of a boundary layer), effective in a small distance from the surface of the sphere and behaving like  $\exp [-(1+i)\sqrt{R_e/2}(R-1)]$ . The nature of boundary

layer solution is essentially the same in hydromagnetics as in a non-magnetic case. The term  $\exp [-(1+i)\sqrt{R_e/2}(R-1)]$  represents a wave of vibrations propagated from the boundary of the sphere with the velocity  $\sqrt{2/R_e}$ , but with rapidly diminishing amplitude, the falling off within a wave length being in the ratio  $e^{-2\pi}$ , or  $1/535$ . The linear magnitude  $\sqrt{2/R_e}$ , important in all problems of oscillations in incompressible viscous fluids, indicates the extent to which the effects of viscosity penetrate into the fluid. In the case of Mercury its value is  $.018 P^{\frac{1}{2}}$  cms, where  $P$  is the period of oscillation in seconds. For liquid Sodium, the corresponding value is  $.079 P^{\frac{1}{2}}$ .

$$(2) \text{ Irrotational } \left( \frac{\sin^2\theta}{R}, \frac{\cos\theta\sin^2\theta}{R^2} \right), \text{ and}$$

$$(3) \text{ Rotational } \left( \frac{\sin^2\theta}{R^3}, \frac{\sin^2\theta\cos^2\theta}{R}, \frac{\sin^2\theta\cos^2\theta}{R^3}, \sin^2\theta\cos\theta \right).$$

In the non-magnetic case, the motion in the flow field is essentially irrotational at large distances, and it is an interesting result that the magnetic field generates vorticity.

In the case of a magnetic field, the term  $\frac{\cos\theta\sin^2\theta}{R^2}$  is current free, while the term  $\cos\theta\sin^2\theta$  is responsible for currents at large distances.

From (49) it follows that for large distances, the disturbance in the present problem differs only in amplitude and phase from the one generated by the oscillation of a sphere in a frictionless fluid in the presence of a magnetic field.

The diagrams 1, 2 and 3 illustrate the effects of various parameters on the flow fields. Figure 1 shows how the drag on the sphere varies as it oscillates. It also shows the influence of conductivity and viscosity on the drag. Figure 2 shows the effect of Reynolds number  $R_e$  on magnetic lines of force. The viscosity tends to contract the lines of force. On the other hand, Figure 3 shows the stretching effect of the magnetic field on the stream lines (compare Figure 3 (a) and (b); (c) and (d)). Viscosity also tends to produce similar effects on the flow (compare Figure 3 (a) and (c); (b) and (d)).

### Appendix

As pointed out earlier, the only effect of rotation in this problem is that azimuthal components of  $\underline{v}$  and  $\underline{H}$  are affected in that they are non-zero here in contrast to the non-rotating case in which  $v_\varphi$  and  $H_\varphi$  are zero.

To determine  $v_{\varphi}$ , we use (21) which gives

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + h^2 \right] v_{\varphi} = \delta R_e v_{or} \quad (A1)$$

and the boundary conditions are

$$v_{\varphi} = 0 \text{ at } R = 1 \text{ and at infinity.} \quad (A2)$$

The solution is (using (26))

$$v_{\varphi} = B_1 x r f_2(hR) + \frac{\delta R_e}{h^2 - k^2} \frac{1}{k^2} \frac{\partial s_0}{\partial r} - \frac{3}{2} \frac{1}{h^2} B h^2 x r f_1(hR) \quad (A3)$$

where the first term is the complementary function and the remaining terms are a particular integral. The constant  $B_1$  can be chosen in such a way that the boundary condition  $v_{\varphi} = 0$  at  $R = 1$  is satisfied.

Next,  $H_{1\varphi}$  can be determined by

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right] H_{1\varphi} = - \frac{\partial v_{\varphi}}{\partial x}, \quad (A4)$$

while inside the sphere

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \right] H_{1\varphi} = 0 \quad (A5)$$

and we require the continuity of  $H_{1\varphi}$  across the surface of the sphere.

The solution is

$$\begin{aligned} H_{1\varphi} = & B_2 \frac{P_1^1(\cos\theta)}{R^2} + B_3 \frac{P_3^1(\cos\theta)}{R^4} + \frac{\delta R_e i}{k^4(h^2 - k^2)} \frac{\partial^2 s_0}{\partial x \partial r} - \\ & - \frac{3i\delta B}{2} \left[ \frac{2}{h^2} \frac{\partial^2}{\partial x^2} + x \frac{\partial}{\partial x} + 1 \right] r f_1(hR) + \\ & + \frac{B_1}{h^2} \frac{\partial}{\partial x} [x r f_2(hR)] , \end{aligned} \quad (A6)$$

and inside the sphere

$$H_{1\varphi} = C_1 R \cdot P_1^1(\cos\theta) + C_2 R^3 \cdot P_3^1(\cos\theta) . \quad (A7)$$

The first two terms in (A6) are the complementary function and the remaining terms are a particular integral. The constants  $B_2$ ,  $B_3$ ,  $C_1$  and  $C_2$  are determined by the requirement that  $H_{1\varphi}$  is continuous at  $R = 1$ .

Finally,  $v_{1\varphi}$  is determined by

$$\left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + h^2 \right] v_{1\varphi} = \beta R_e v_{\alpha\varphi} + \delta R_e v_{1r} \quad (A8)$$

and the boundary conditions  $v_{1\varphi} = 0$  at  $R = 1$  and at infinity. (A9)

The solution is

$$\begin{aligned} v_{1\varphi} = & G_1 R^2 \cdot P_2^1(\cos\theta) f_2(hR) + G_2 R^4 \cdot P_4^1(\cos\theta) f_4(hR) + \\ & + \delta R_e \left[ - \frac{\beta}{2k^2} \left\{ \frac{4h^2 - 2k^2}{(h^2 - k^2)^2} \frac{\partial s_0}{\partial r} + \frac{x}{h^2 - k^2} \frac{\partial^2 s_0}{\partial x \partial r} - \right. \right. \\ & - \frac{4}{(h^2 - k^2)^2} \frac{\partial^3 s_0}{\partial r \partial x^2} - \frac{A}{h^2 - k^2} \left( x \frac{\partial f_0(kR)}{\partial r} - \right. \\ & \left. \left. - \frac{2}{h^2 - k^2} \frac{\partial^2 f_0(kR)}{\partial r \partial x} \right) \right\} - \frac{3}{8} B \beta i x r \left\{ f_1(hR) - \right. \\ & \left. - f_0(hR) + h^2 x^2 f_2(hR) \right\} + \frac{i}{k^2 (h^2 - k^2)} \frac{\partial s_1^c}{\partial r} - \\ & - \frac{3}{2h^2} (h^2 F_1 - 4F_2) x r f_1(hR) + \\ & + \frac{15}{4} F_2 x r f_3(hR) (-4x^2 + 3r^2) \left. \right] - \frac{B_1 \beta i}{2} x r f_1(hR) \end{aligned} \quad (A10)$$

where the first two terms are the complementary function and the remaining are a particular integral. The constants  $G_1$  and  $G_2$  are determined by the requirement that  $v_{1\varphi} = 0$  at  $R = 1$ .

It is clear from the above solution that there will be no contribution of Maxwell and Viscous stresses (produced due to rotation) on the drag—a result in agreement with our previous conclusions (see Singh, 1964 [12]).

Since the wave-like terms  $f_n(\zeta)$  are damped heavily, it follows that in non-rotating compressible case, the flow will be essentially current free and irrotational at large distances. Our solution above shows that the effect of rotation will be to produce vorticity and current which again agrees with our conclusions in [12].

Concluding, it should be noted that the nature of wave propagation remains essentially similar in both rotating and non-rotating cases.



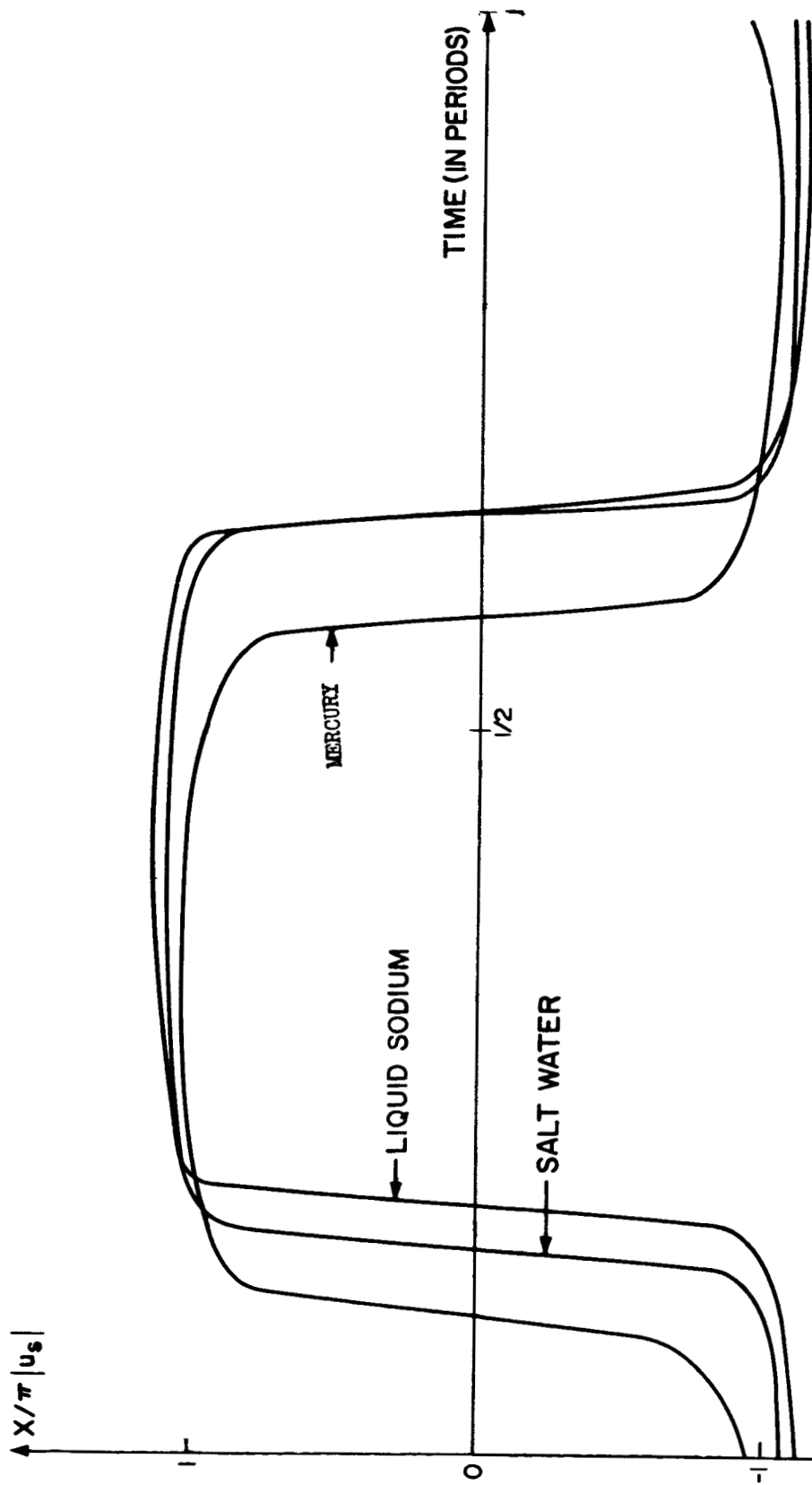
Acknowledgements

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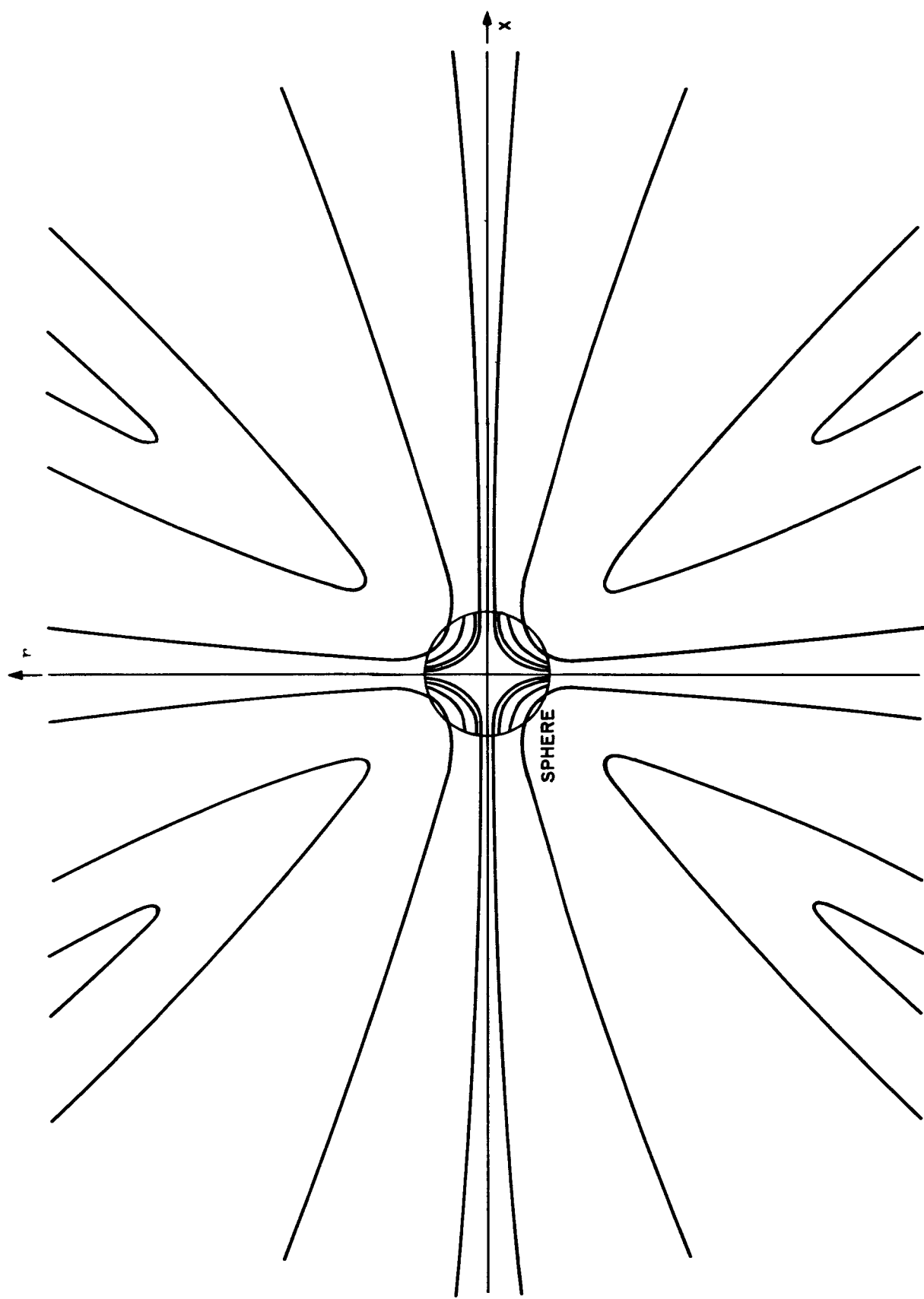
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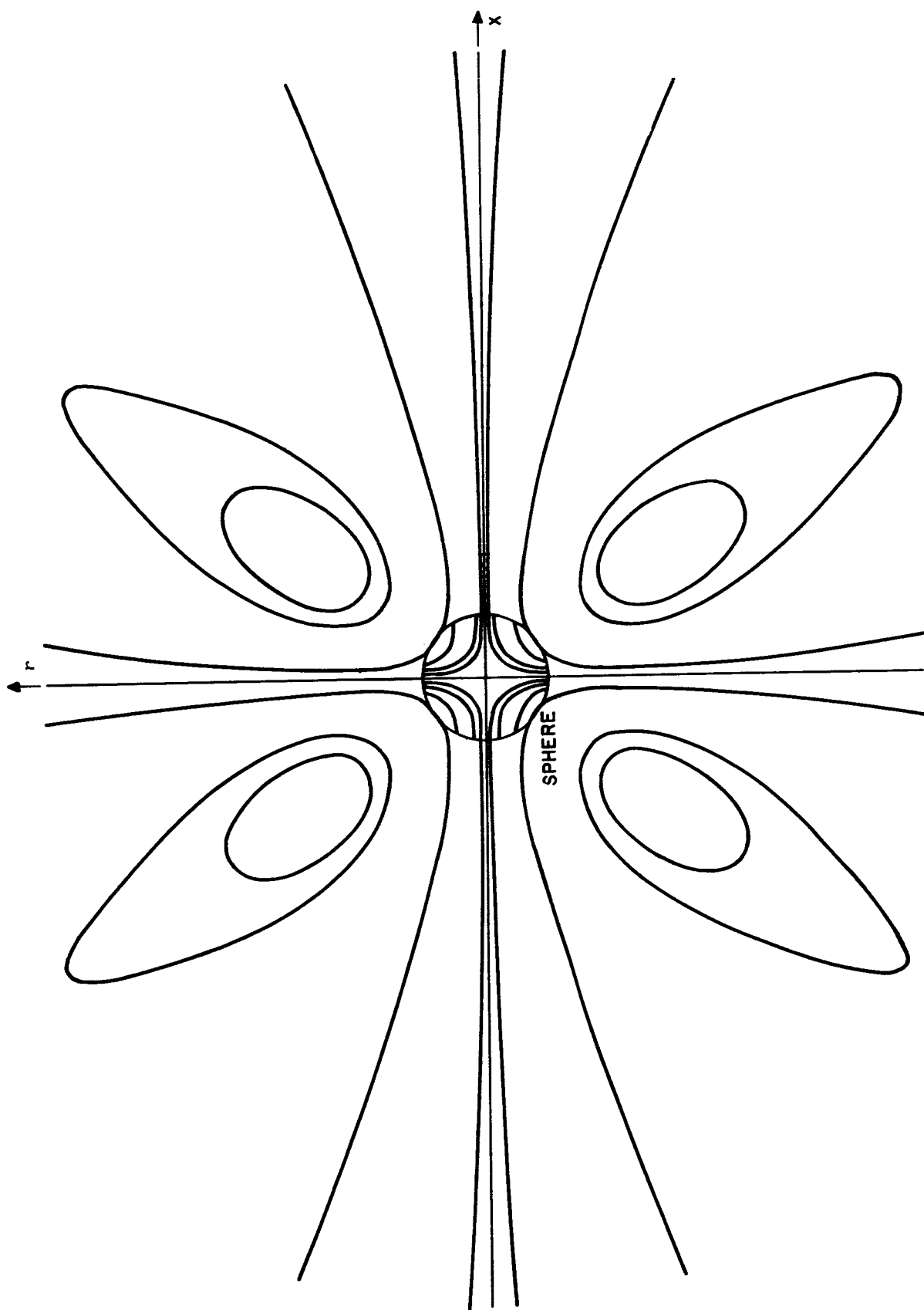
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DRAG VS. TIME  
FIG. 1

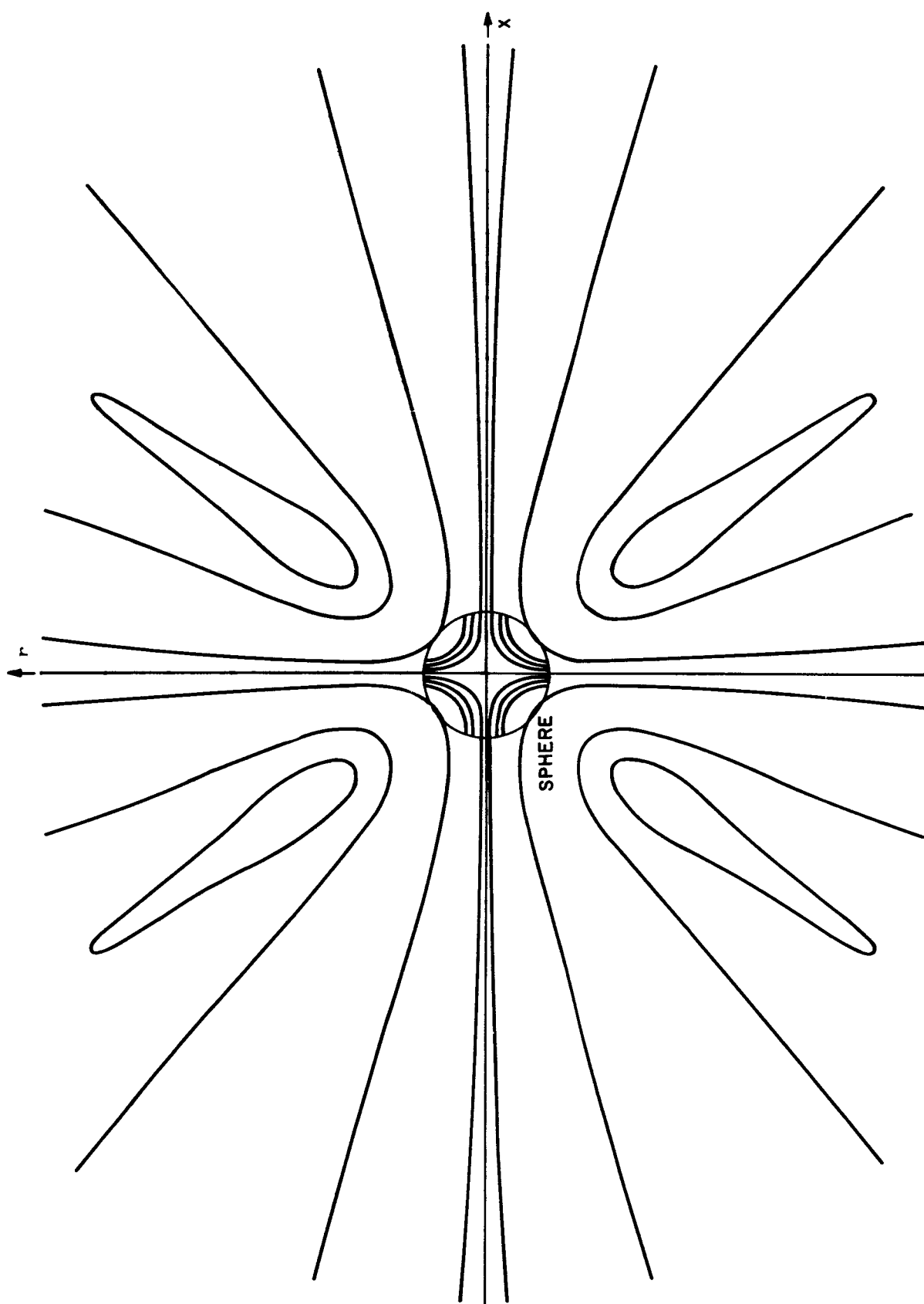


MAGNETIC LINES OF FORCE (PERTURBED) AT .95th OF THE OSCILLATION  
(MERCURY)  
FIG. 2(d)



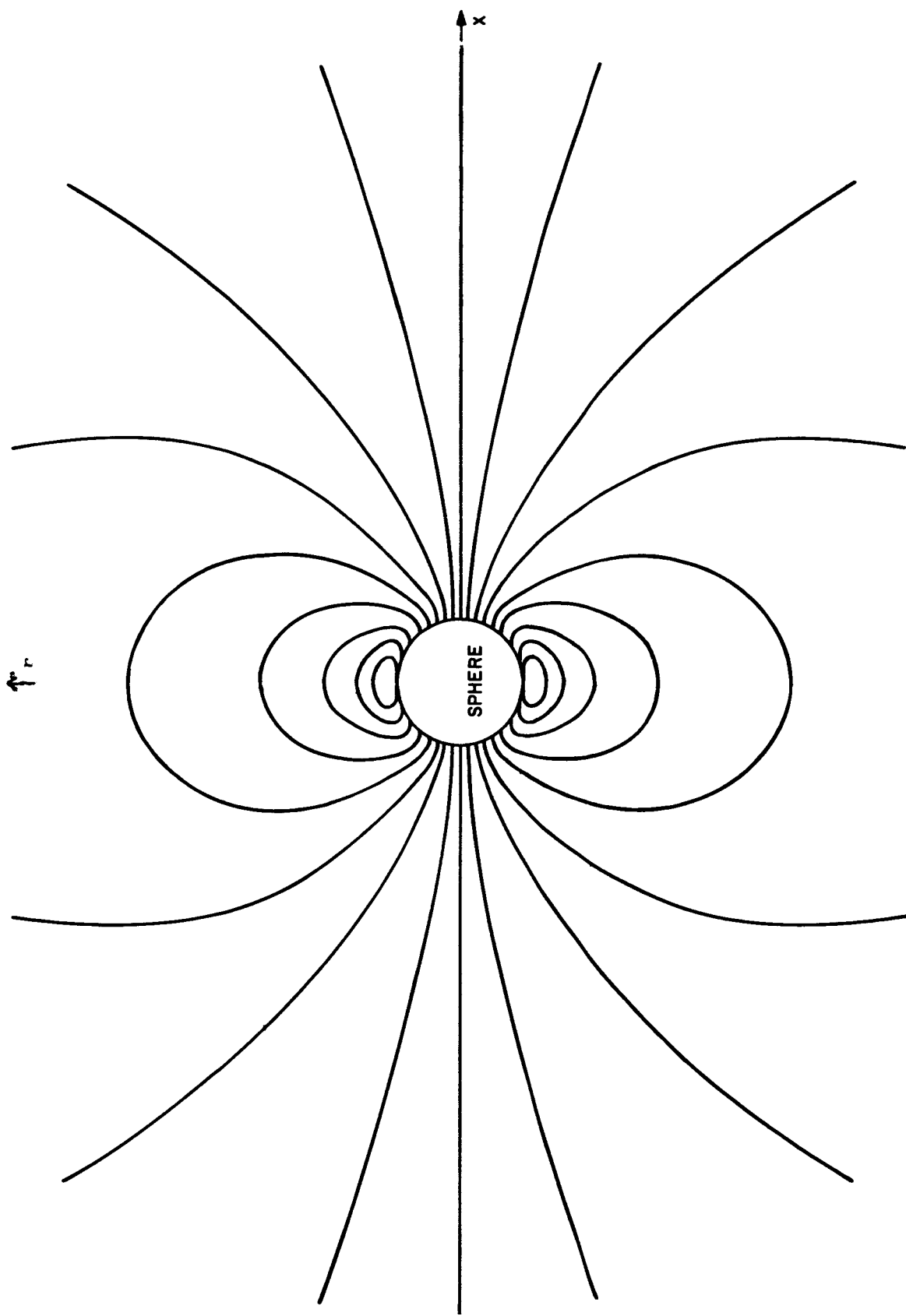
MAGNETIC LINES OF FORCE (PERTURBED) AT .95th OF THE OSCILLATION  
(LIQUID SODIUM)

FIG. 2(b)



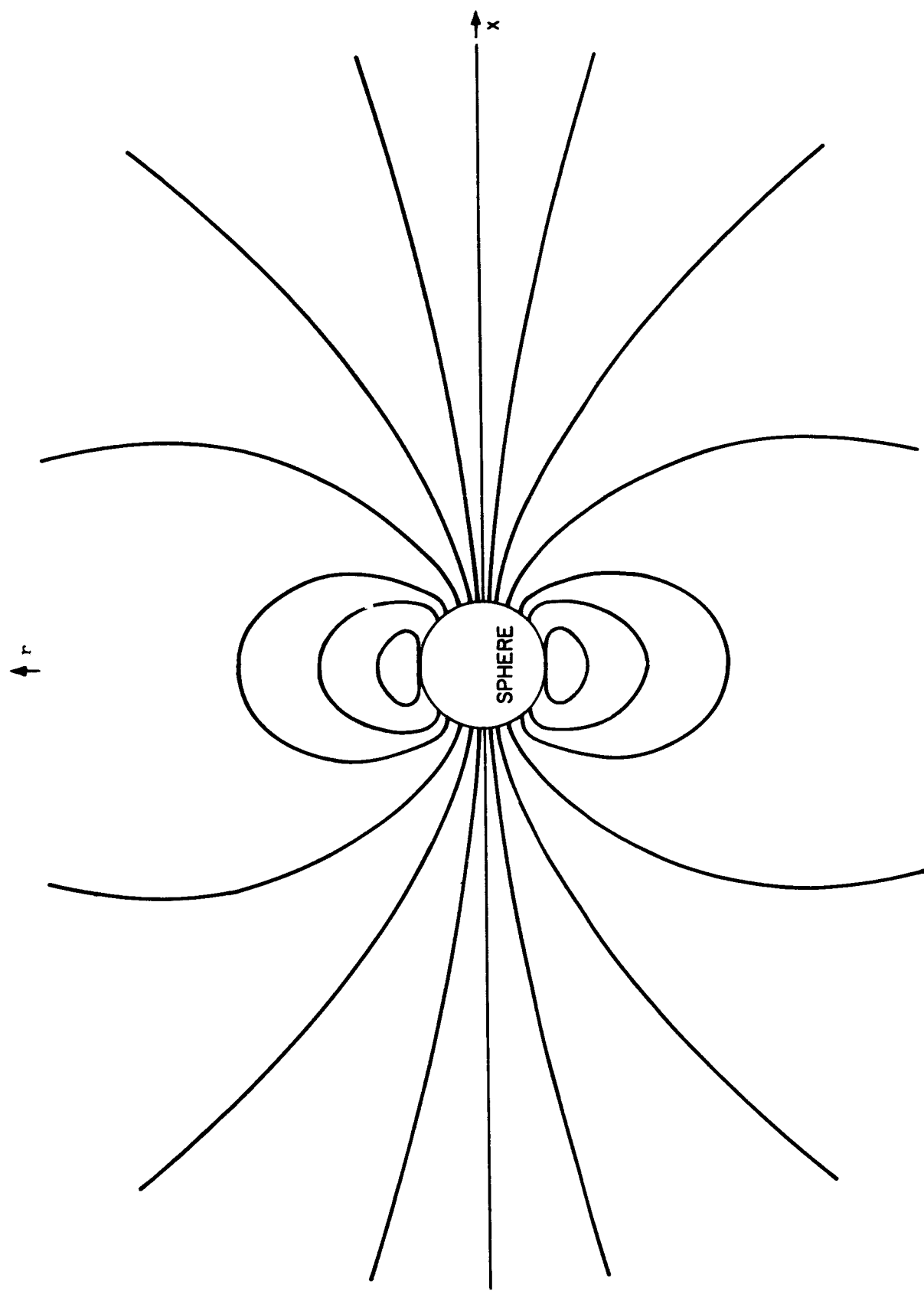
MAGNETIC LINES OF FORCE (PERTURBED) AT .95th OF THE OSCILLATION  
(SALT WATER)

FIG. 2(c)



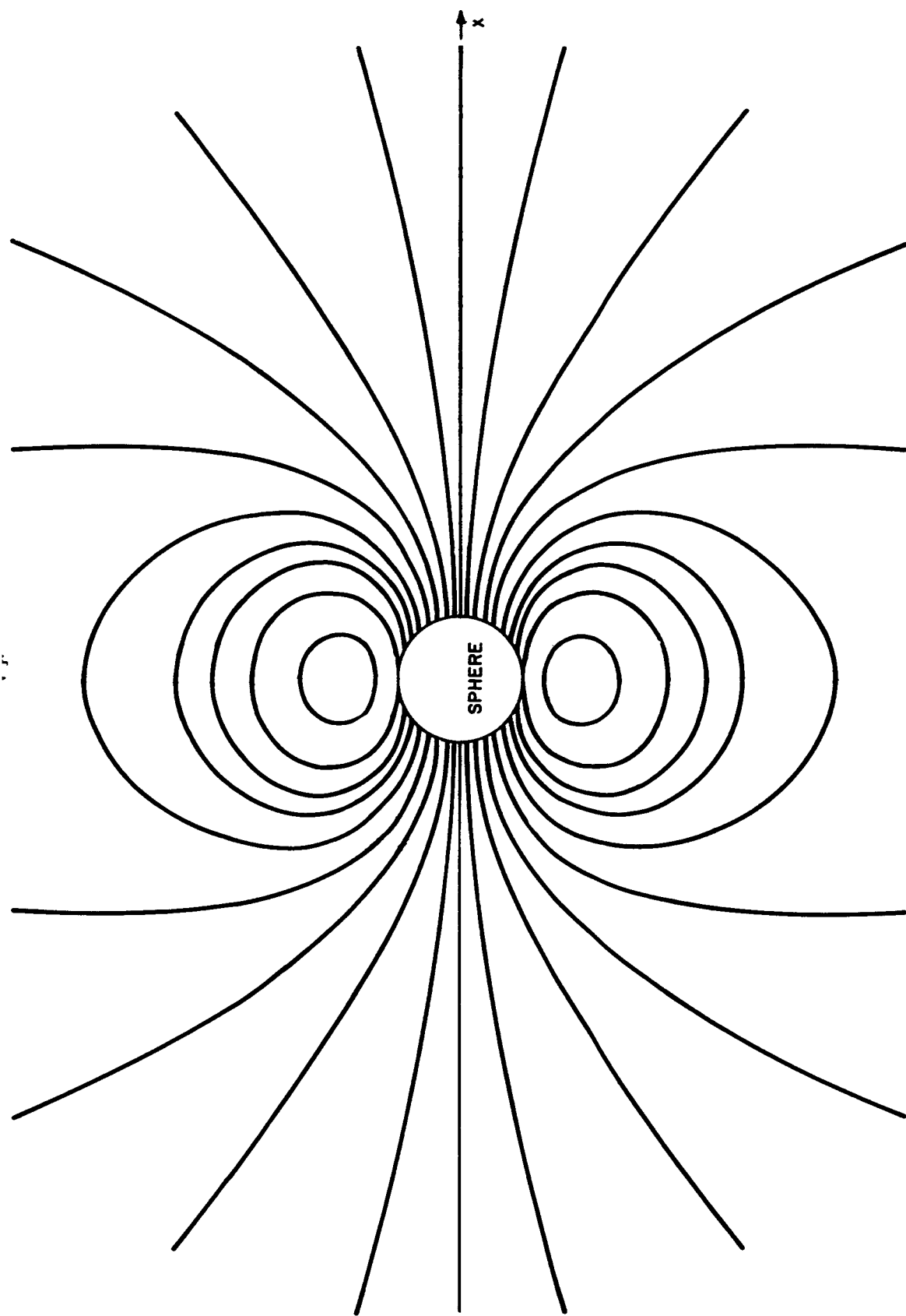
STREAMLINES PATTERN (NON-MAGNETIC CASE) AT  
THE COMPLETION OF AN OSCILLATION  
(MERCURY)  
FIG. 3 (a)





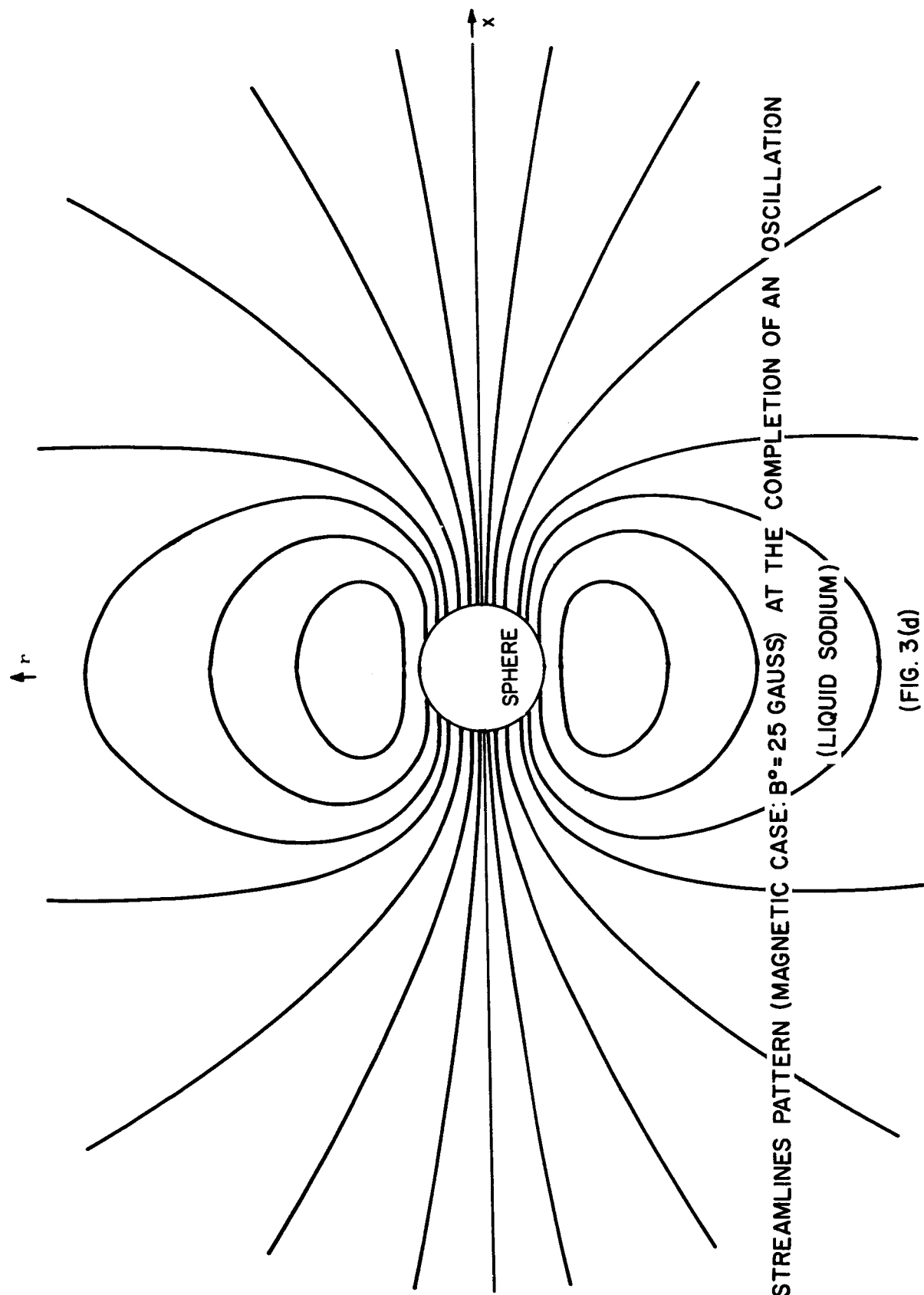
STREAMLINES PATTERN (MAGNETIC CASE:  $B^0 = 25$  GAUSS)  
AT THE COMPLETION OF AN OSCILLATION  
(MERCURY)

FIG. 3 (b)



STREAMLINES PATTERN (NON-MAGNETIC CASE)  
AT THE COMPLETION OF AN OSCILLATION  
(LIQUID SODIUM)

FIG. 3(c)



(FIG. 3(d))